Simple Origami Models with Boundaries on the Edges of Platonic Solids

R. Geretschläger

1 Introduction

When a unit square is folded to create an origami model in three-dimensional space, the edge of the paper forms a closed curve in space with a total length equal to four units. We know that the model fully determines its edge, but it can be interesting to derive a model when its boundary curve (or polyline for classic models utilizing only straight folds) is given. In other words, we would like to know to what extent the edge determines the model.

This is quite a general question, and a complete solution is more of a theoretical than practical nature. Reconstructing a possible origami model from known properties of the edges can be restricted to various specific types of closed curves or polylines, however, and such restrictions can yield models with some very interesting special properties.

In this paper, I present some results of such reconstructions pertaining to boundaries composed wholly of edges of platonic solids. I will refer to the resulting origami models as *Platonic-Edge Origamis*, or PEOs for short.

I will be using some of the usual shortcuts in terminology, making some useful simplifications identifying objects that can exist in the physical world with theoretical contructs in ways that I can only hope that no one will actively object to. A "sheet of paper" will be assumed to be a plane (and thus infinite) embedded in $\mathbb{R}^3$ (or, equivalently, in euclidean 3-space E^3). A "unit square" will be a square section of such a plane with sides of unit length. An "origami model" (or simply an "origami") will be a connected collection of plane objects that can result from the unit square by folding procedures. The plane sections of the origami model will be referred to as "facets" of the origami model, and its edges will be referred to as "creases", if they are not part of the bounding polyline. This means that the term "edge" will be reserved for a line segment, which is a section of the bounding polyline.

I will also assume that the origami models are, in general, not flat, but rather of a general three-dimensional nature. I will, however assume that all facets of all considered origamis are planar (convex) polygons.

2 Restrictions on Polylines Composed of Edges of Platonic Solids

There are several a priori restrictions to polylines that could possibly be boundaries of origami models folded from a unit square, and such restrictions are considered in [1].

Some of these are quite obvious. For instance, not only must the total length of the polyline be equal to four units, the fact that the edges of the unit square are at right angles to each other has an immediate consequence. This implies that the polyline must consist of four sections, each of unit length, with the angles in the corners joining the sections each no larger than a right angle. The latter will, for instance, have consequences with respect to the search for certain types of PEOs on the dodecahedron, since the angles between any two adjoining edges of a dodecahedron are always equal to the interior angle of a regular pentagon, i.e. 108°, and thus larger than a right angle.

For the purposes of this paper, we will call a polyline *possible*, if it fulfills all the requirements of the edges of an origami model. In this context, we will not need to worry about the precise restrictions this involves, but it will be useful to be able to refer to "possible" polylines, to distinguish them from general polylines in E^3.

An important restriction for finding PEOs results from the fact that the polyline must be either closed or contain certain edges multiple times. The first option means that there must exist a closed path on the Platonic solid, composed of edges, which may be traversed once or several times. If we do not insist on a closed polyline in space, the edge of the origami model must nevertheless be "closed" in the sense that we can follow the path all the way around the edges of the folding square without ever departing from the edges, and returning to the point in which we started. An example of such a path is shown in Figure 1.

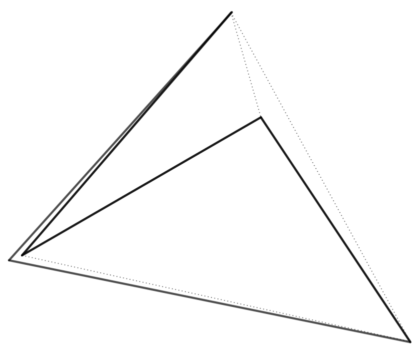

Figure 1: a "closed" path on a non-closed group of tetrahedron edges

At this juncture, we have not yet decided whether we will be able to find PEOs of specific types with such a boundary, of course, but we can consider it "closed" from the point of view of the edge of a possible model, even though the four tetrahedron edges shown do not actually form a closed polyline.

3 Categories of Platonic.Edge Origamis

There are several categories of origamis we can search for in the given context. For instance, we can consider the following types:

1. (A): an origami whose edge is composed of four line segments of unit length, which correspond to four edges of a certain Platonic solid
2. (B): an origami whose edge is a closed polyline, all segments of which correspond to edges of a certain Platonic solid, without any edge being covered more than once
3. (C): an origami whose edge is a closed polyline, all segments of which correspond to edges of a certain Platonic solid, without restrictions on an edge being covered more than once
4. (D): an origami whose edge is composed of line segments which cover all edges of a certain Platonic solid
5. (E): an origami whose edge is composed of line segments of the longest possible length while corresponding to a specific number of edges of a certain Platonic solid
6. (F): an origami whose edge is composed of line segments corresponding to a specific number of edges of a certain Platonic solid without regard to their length

It is clear that this list is not complete, but it gives us a nice illustration of the number of possible types of origamis we can seek to create with their edges corresponding exclusively to edges of Platonic solids. We will use some of the designations from this list for the types of origamis in the text to follow.

Even a cursory glance at this list gives us the feeling that PEOs of some of these types will be of more interest than others. It would seem that type (A) or type (E) PEOs would be more intriguing than type (C), for instance, although such an initial impression may be quite deceptive. The taxonomy will prove to be quite useful, however, when we want to refer to models with certain specific properties.

4 One-edge Platonic-Edge Origamis and Models Derived from Them

With these preliminary remarks out of the way, we are now ready to face the challenge of finding concrete Platonic-Edge Origamis with various properties. The simplest solution to the general problem of finding an origami model whose edges all lie on the edges of a given Platonic solid is the unit square itself, of course. It is already the face of a cube, or alternatively the central planar intersection of a regular octahedron, as illustrated in Figure 2.

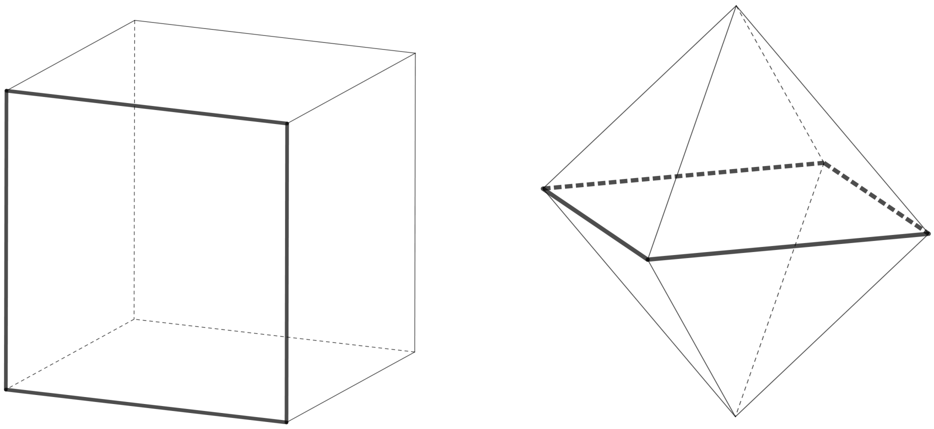

Figure 2: the trivial PEO

We therefore obtain our first PEO model without any folding at all.

This observation starts us on our journey, but we find a solution almost as trivial by folding the unit square in such a way that the edges all coincide. Two simple ways to do this are illustrated in Figures 3 and 4. In the left-hand figures, we first fold the unit square diagonally and then fold the resulting triangle in half again. In the right-hand figures, we see the classic water-bomb base.

In each of these two cases, the edge of the resulting origami is a line segment of unit length, covered four times. This line segment can be interpreted as an edge of any Platonic solid, of course. Since such an edge can never be longer than one unit, these are extremely basic examples of PEOs of type (E).

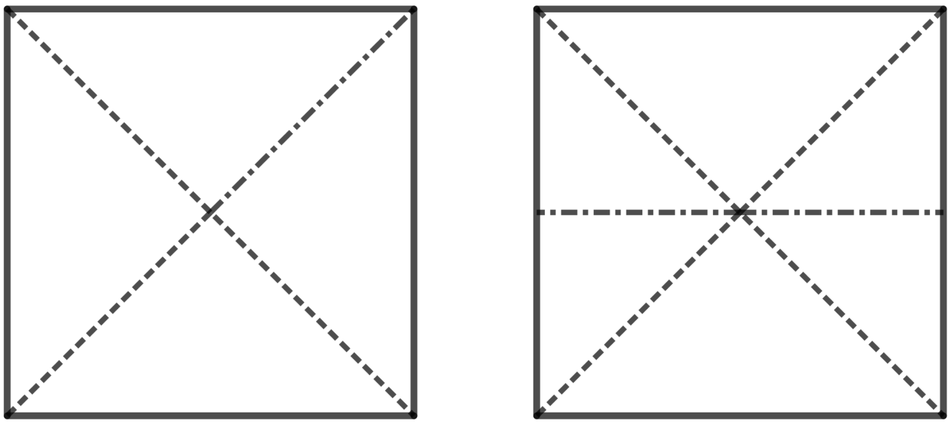

Figure 3: one edge folding patterns

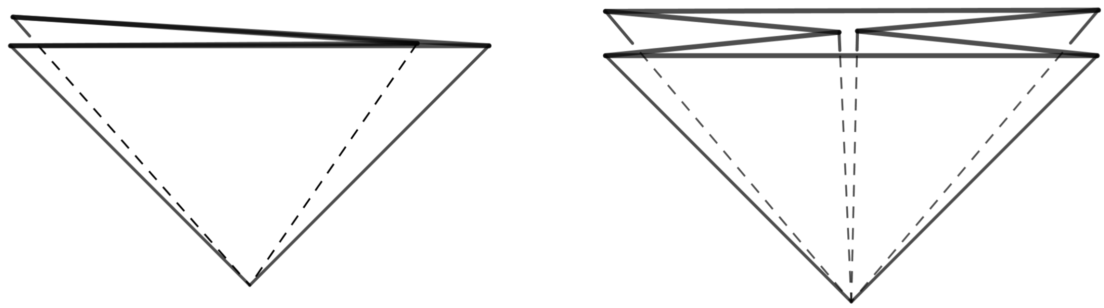

Figure 4: one edge models

Since we can fold the triangles of these models in zig-zags, as shown in 5, the folding square can, in principle, be reduced to a thin strip that we can treat as a line segment, which can then be folded to cover any polyline we wish.

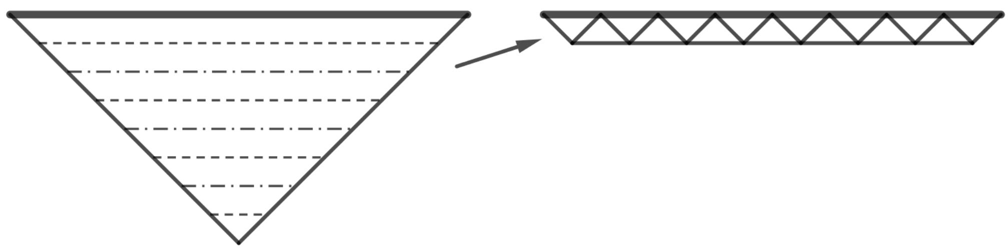

Figure 5: zig-zag

Of course, this is a purely theoretical option if we need to zig-zag any more than shown here, as the thickness of the paper will start to play a big role in limiting our folding options. Also, if we want to create something like a PEO whose edge covers all 30 edges of an icosahedron, the length of each individual edge will be very small (less than $\frac{1}{30}$ of the unit length in this specific case), which creates another physical limit to what can actually be done with real paper.

If we take a closer look at the water-bomb base, we find that this simple starting point yields the solutions to some intriguing problems that we can pose with respect to the edges of a Platonic Solid. Two nice examples are shown in Figure 6.

On the left of this figure, we have marked four edges of a cube, which can be interpreted as looking a bit like a capital M. We set ourselves the problem of folding a unit square in such a way that the edge of the square comes to lie exclusively on the line segments of this M, covering it completely. Similarly, on the right, we have marked six edges, including the four already marked on the left, but now also adding the other two sides of the top-most square of the cube. Again, we aim to fold a unit square with the analogous goal of covering these segments.

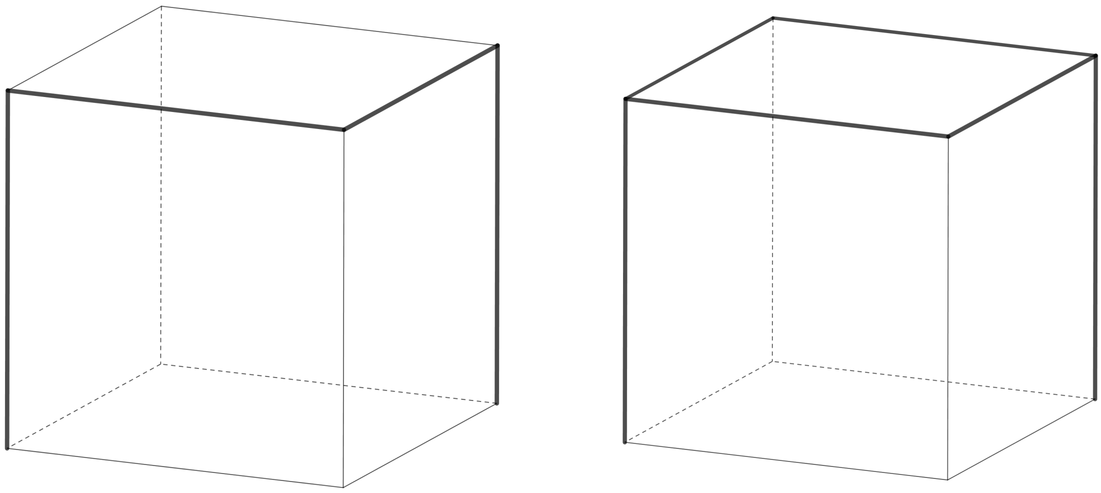

Figure 6: M on cube edges

In order to solve both of these challenges, one option is to fold the water-bomb base in the middle as illustrated in Figure 7.

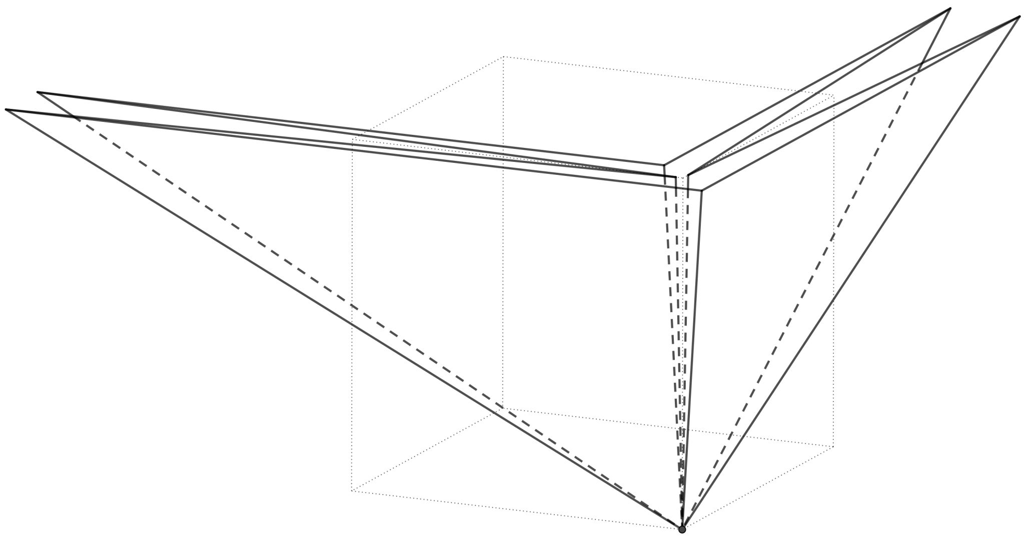

Figure 7: first step

We can flatten this figure and treat it as if it were a triangle, as shown in Figure 8. A solution for the M will result by simply folding the overhanging sections of the triangle down, and a solution for the six-edged configuration will result by folding the front flaps down in the same way, and simultaneously folding the rear flaps back.

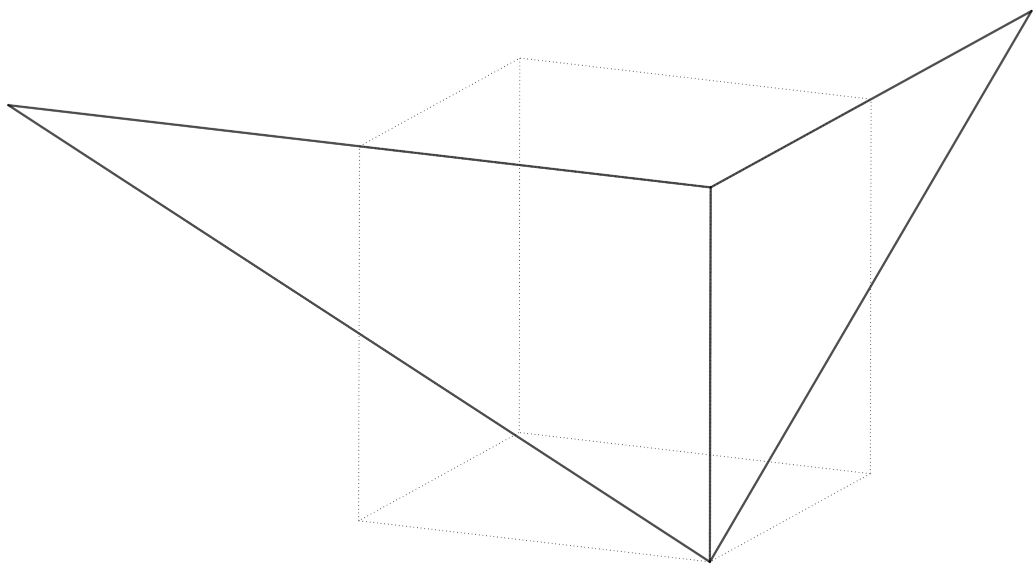

Figure 8: first step pushed together

We first consider the solution to the M configuration. As we see in Figure 9, two valley folds are required to bring the overhanging parts of the edge to conicide with the vertical cube edges, resulting in the situation illustrated in Figure 10.

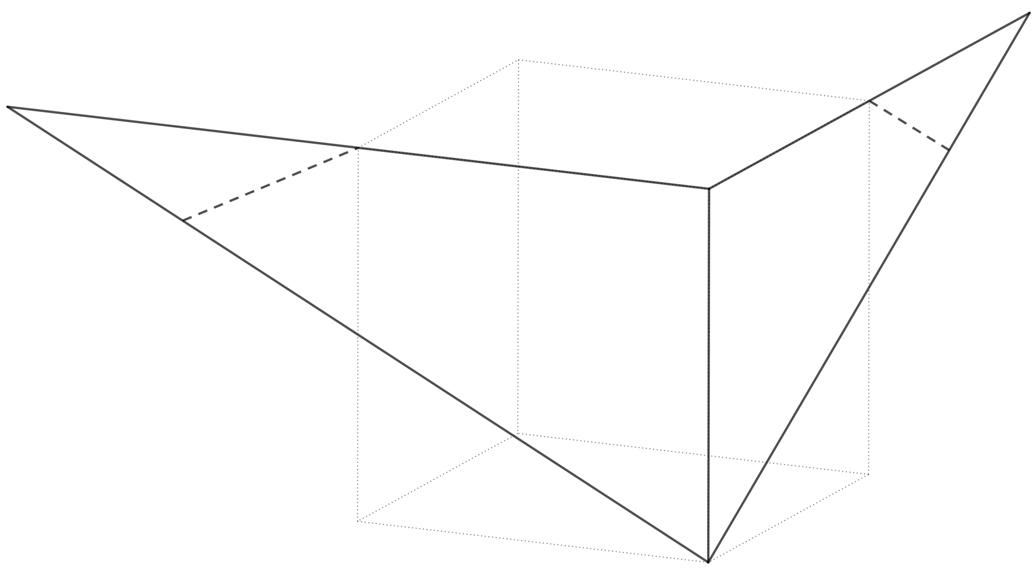

Figure 9: first step pushed together with folds

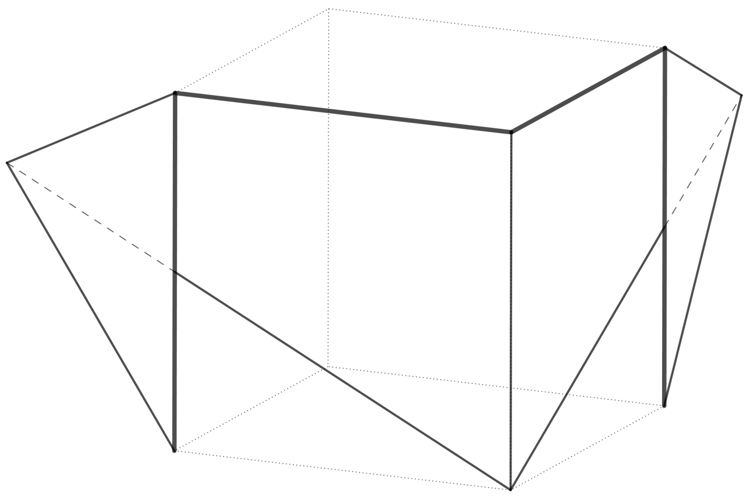

Figure 10: PEO with its edge on the M

The solution to the right-hand configuration is then quite similar. As illustrated in Figure 11, two valley folds in the front flaps again bring the overhanging front parts of the edge to conicide with the vertical cube edges, while two mountain vertical folds in the back flaps bring the overhanging back parts of the edge to coincide with the two remaining cube edges. This results in the situation illustrated in Figure 12.

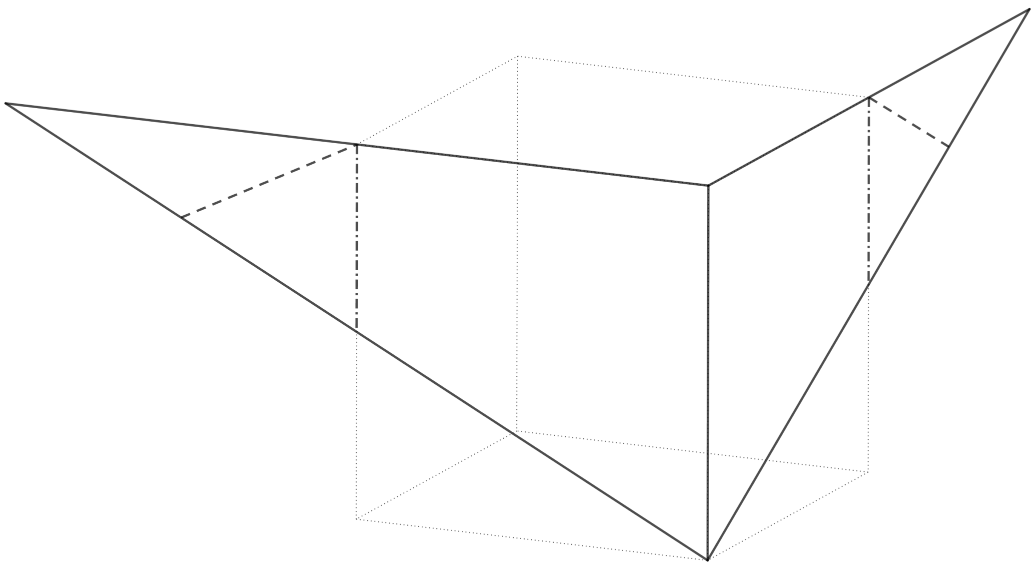

Figure 11: first step pushed together with folds in both directions

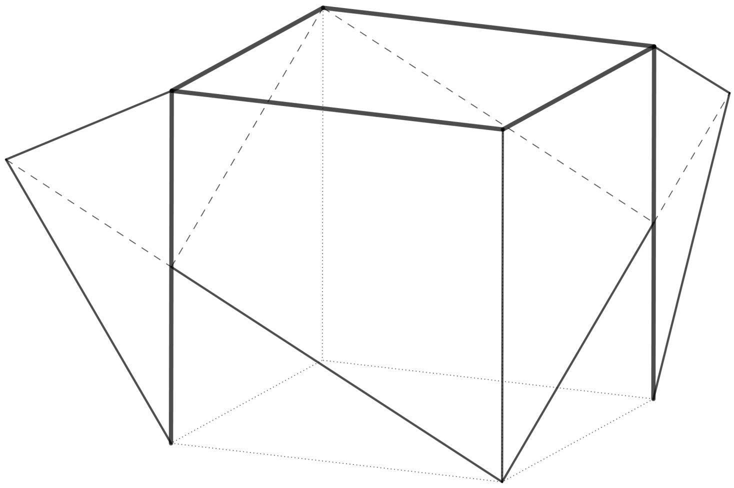

Figure 12: the second PEO

5 Type (A) PEOs

In this section, we will take a look at the matter of finding Type (A) PEOs, i.e. origamis, whose edges are composed of four line segments of unit length, corresponding to four edges of a specific Platonic solid. The trivial PEOs we have already seen in Figure 2 are both examples.

In order to find such an origami, we must first identify a closed 4-edge path on a Platonic solid, and then decide how to fold the unit square such that its edge coincides with this path, under the assumption that the edges of the solid are all of unit length. We have already mentioned the trivial PEO covering the face of a cube, and this is the only possible Type (A) PEO with edges on a cube, as the sides of a square cube face form the only possible closed polyline composed of four edges of the cube. We also immediately note that there can be no Type (A) PEO with edges on a dodecahedron, as there is no closed polyline composed of four edges of a dodecahedron.

We can therefore restrict our attention in the following to Type (A) PEOs on the Platonic solids with triangular faces.

As we see in Figure 13, there can also be no Type (A) PEO with its edges on an icosahedron. The only possible 4-edge path on the edges of an icosahedron is composed of edges of two adjacent triangles, as illustrated here by the quadrilateral $ABCP$, whose edges lie on the adjacent triangles ABP and CPB. If it were possible to fold a unit square in such a way that its edges come to coincide with the edges of $ABCP$, two diagonally opposite vertices of the folding square would come to lie in points A and C. However, AC is a diagonal of a regular pentagon $ABCDE$, whose sides are edges of the icosahedron, and thus have unit length. The length of AC is therefore equal to $\frac{\sqrt{5}+1}{2}$. The diagonal of the unit square, however, has a length of $\sqrt{2}$, and since $\frac{\sqrt{5}+1}{2} > 1.6 > \sqrt{2}$ the existence of such a model would imply that it was possible to fold the unit square in such a way that the distance between diagonally opposite vertices grows larger during the folding process, which is clearly impossible.

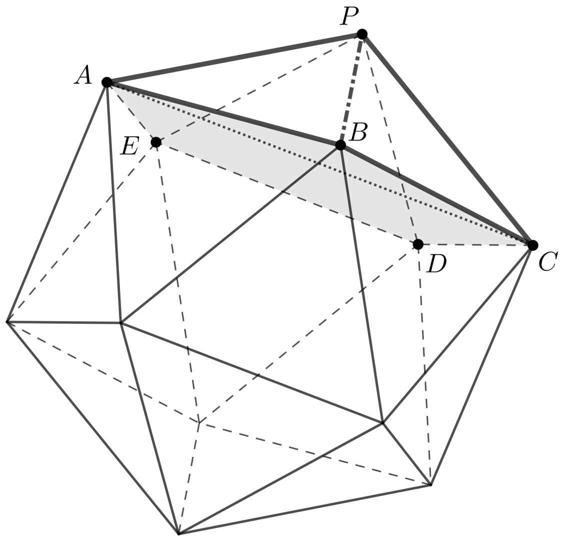

Figure 13: 4-edge path on icosahedron edges

Next, we consider Type (A) PEOs with edges on an octahedron. In Figure 14, we see a similar situation to the one previously considered for the icosahedron.

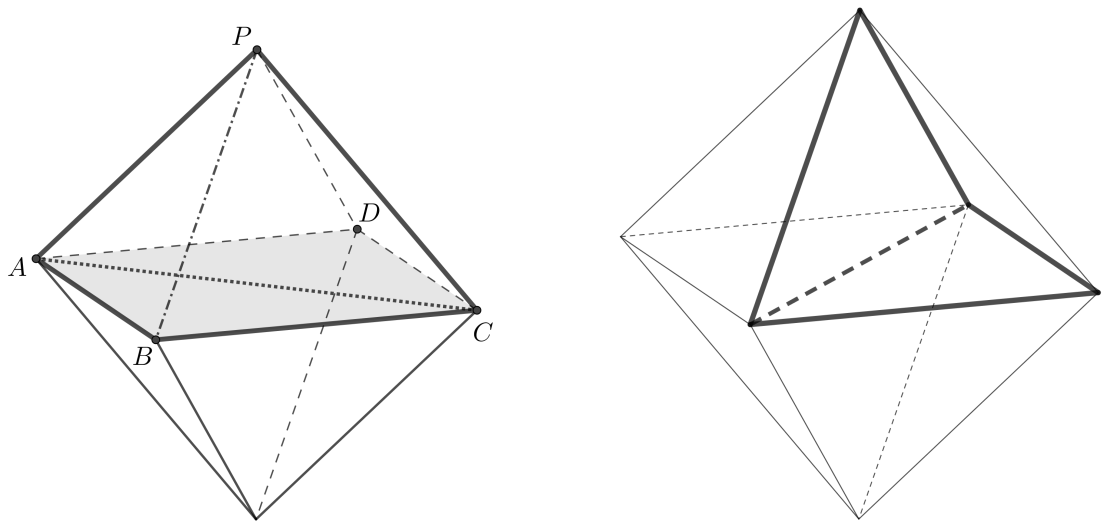

Figure 14: 4-edge path on octahedron edges

Other than the square we previously encountered in the trivial solution shown in Figure 2, the only possible 4-edge path on the edges of an octahedron is composed of edges of two adjacent triangles. In this case, we again have a quadrilateral *ABCP*, whose edges lie on the adjacent triangles *ABP* and *CPB*. If we wish to fold a unit square in such a way that its edges coincide with the edges of *ABCP*, two diagonally opposite vertices of the folding square must come to lie in points *A* and *C*. Since *ABCD* is a square, the length of *AC* is equal to the length $\sqrt{2}$ of the diagonal of the unit square, and must therefore remain unchanged by the folding procedure. This implies that triangles *ABC* and *ACP* are each half of the folding square, which yields the position shown on the right of the figure (rotated by 90°). In this way, we see that a Type (A) PEO is obtained by folding the square at a right angle along one of its diagonals.

So far, we have determined that there are no Type (A) PEOs on either the dodecahedron or the icosahedron, only the trivial PEO on the cube, and the trivial PEO as well as one other quite simple one on the octahedron. What remains is to find a Type (A) PEO on the regular tetrahedron. Once again, we require a closed 4-edge path on this solid, and the only such path possible is again composed of edges of two adjacent triangles, as shown in Figure 15.

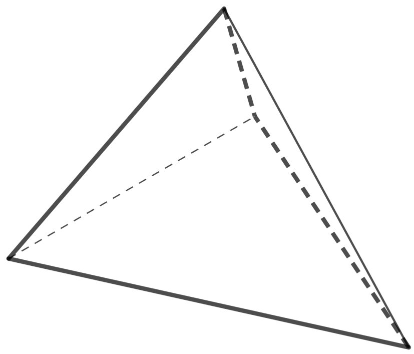

Figure 15: possible 4-path on a tetrahedron

Folding the square to bring the edges into this position is not difficult. We can simply fold the square as shown in Figure 16, noting that the inner point of the square in which the four folds meet is the point dividing the diagonal of the square in the ratio 1 : 3. In order to make clear why this yields a PEO with the required properties, we consider the following steps.

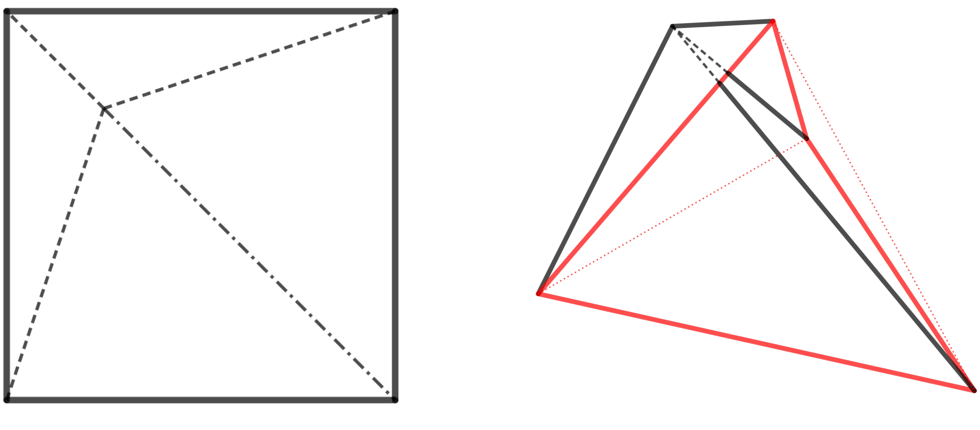

Figure 16: folding pattern for tetrahedron PEO and resulting model

In Figure 17, we have folded the square on its diagonal and placed the result in a system of coordinates, such that two adjacent sides of the square come to lie in AB and BC with an equilateral triangle ABC. As the sides of the square have unit length, this means that the coordinates of these three points are $A(\frac{1}{2},0,0)$, $B(0,\frac{\sqrt{3}}{2},0)$ and $C(-\frac{1}{2},0,0)$.

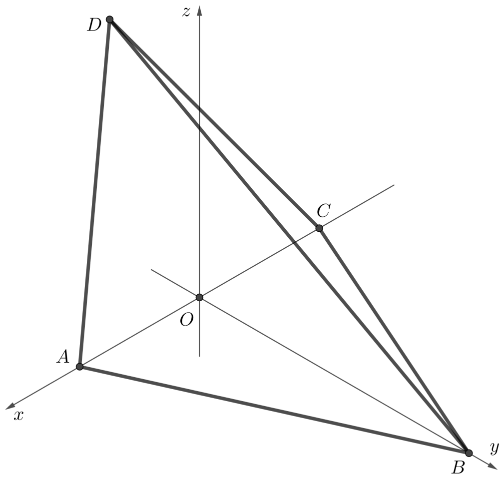

Figure 17: tetrahedron step 1

We note that triangle *ACD* is then also equilateral, since the sides of the folding square are all of the same length. Next, in Figure 18, we introduce the *xz*-coordinate plane. This plane intersects the line *BD* in a point we name *E*.

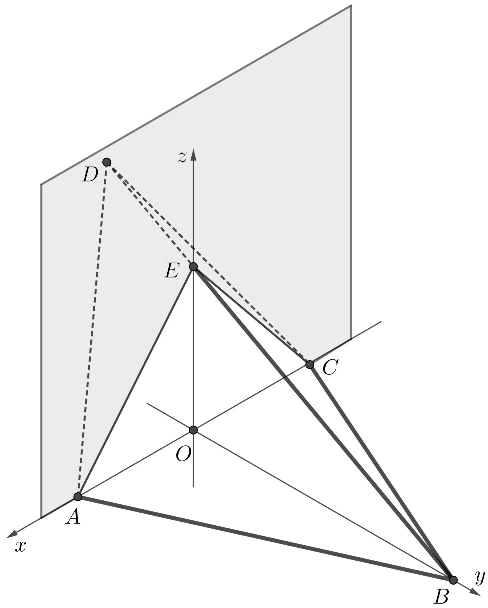

Figure 18: tetrahedron step 2

We will show that this point *E* is the point dividing *BD* in the ratio $1:3$. If we now reflect the part of the model with negative *y*-coordinates on the *xz*-coordinate plane as shown in Figure 19, point *D* goes to the symmetric point *P*, and we obtain the model shown in Figure 16. By symmetry, it is clear that triangle *PAC* is congruent to triangle *DAC*, which we have already noted to be equilateral.

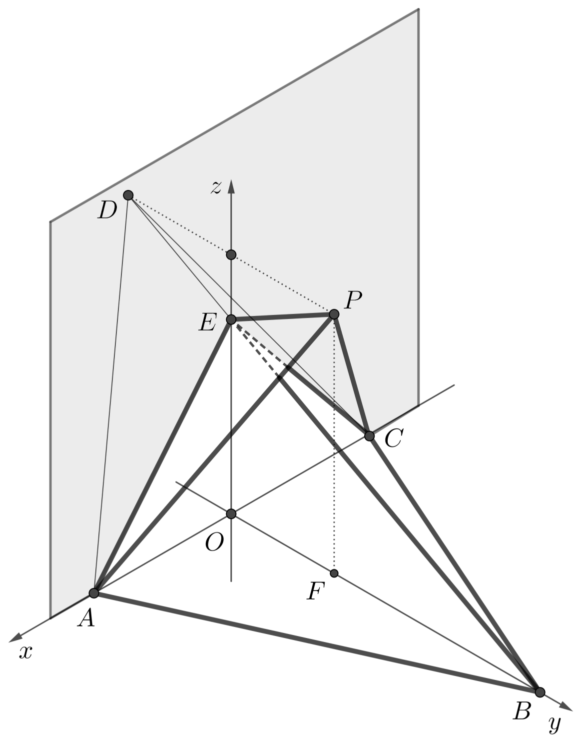

Figure 19: tetrahedron step 3

In order to show that *AB*, *BC*, *CP* and *PA* are indeed edges of a common regular tetrahedron, it only remains to show that $OF = \frac{1}{3} \cdot OB$, with *F* denoting the foot of point *P* on the plane of triangle *ABC* (i.e. the *xy*-coordinate plane).

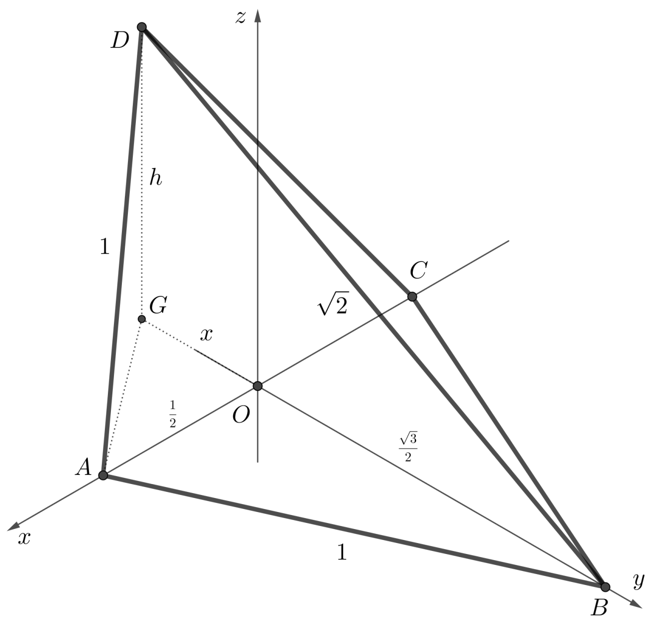

Figure 20: calculating distances

In order to do this, we introduce notation as shown in Figure 20. Here, point G is the foot of point D on the xy-coordinate plane. (We note that this point G is symmetric to point F as introduced in Figure 19 with respect to the x-axis, because of the symmetry with respect to the xz-plane.) We define $OG = x$ and $GD = h$. Since quadrilateral $ABCD$ results by placement of a unit square folded along its diagonal, we have $AB = BC = CD = DA = 1$ and $BD = \sqrt{2}$ Also, we have already noted that the coordinates of A are $(\frac{1}{2}, 0, 0)$, which gives us $AO = \frac{1}{2}$. We are now ready for a little bit of calculation.

Since BDG is a right triangle, we have $BG^2 + GD^2 = BD^2$, or

$$\left(\frac{\sqrt{3}}{2} + x\right)^2 + h^2 = (\sqrt{2})^2,$$

which is equivalent to

$$\frac{3}{4} + \sqrt{3} \cdot x + x^2 + h^2 = 2.$$

Also, in right triangles AOG and AGD, we have $AO^2 + OG^2 = AG^2$ or $\frac{1}{4} + x^2 = AG^2$ and $AG^2 + GD^2 = AD^2$, or

$$\left(\frac{1}{4} + x^2\right) + h^2 = 1.$$

Subtracting the second equation from the first gives us

$$\frac{1}{2} + \sqrt{3} \cdot x = 1,$$

which is equivalent to $\sqrt{3} \cdot x = \frac{1}{2}$ or $x = \frac{\sqrt{3}}{6}$. We see that

$$OF = OG = x = \frac{\sqrt{3}}{6} = \frac{1}{3} \cdot \frac{\sqrt{3}}{2} = \frac{1}{3} \cdot OB$$

does indeed hold as required, and the model resulting in Figure 16 is indeed a Type (A) PEO with edges on the regular tetrahedron, as claimed.

6 Some Intriguing Type (E) and (F) PEOs

Now that we have an overview over Type (A) models, which all have edges of unit length, we can begin our search for models whose edges correspond to more than four edges of a Platonic solid.

6.1 6 Edges on a Cube

If we wish to find a PEO with its edges on exactly six edges of a cube, we must first determine a closed path composed of six cube edges. One such polyline is shown in Figure 21. On the

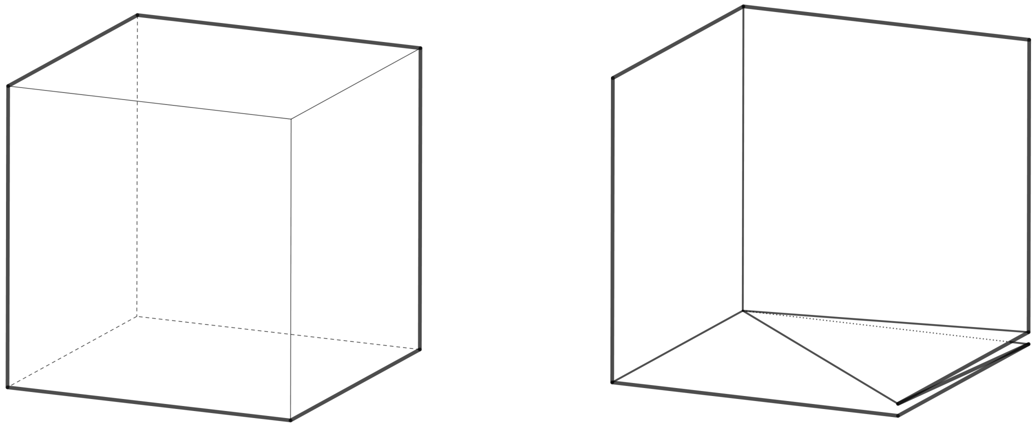

Figure 21: a PEO covering six edges of a cube

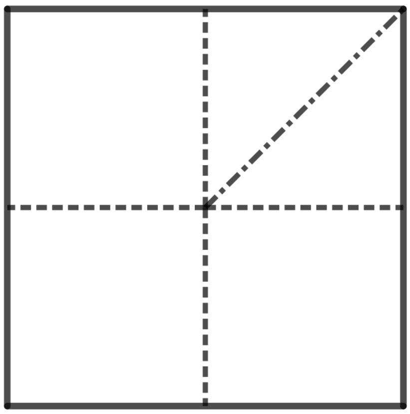

Figure 22: folding pattern

left, we see a path leading around the "outside" of the cube, and to its right, we see a model of an associated PEO. Underneath these two, we see the folding pattern that creates this model.

We note that each of the edges is covered by half of an edge of the folding square, with one of the edges being covered three times in a zig-zag fashion.

It is now interesting to note that another PEO, also covering six edges of the cube, but along a different path, can immediately be derived from this one. We see this illustrated in Figure 23. Here, the polyline is the path around two adjoining faces of the cube, as we can see in the left-hand picture. The model covering this path results from the previous one by simply folding over one corner, as shown in the right-hand picture. Again, we also see the associated folding pattern in the picture below.

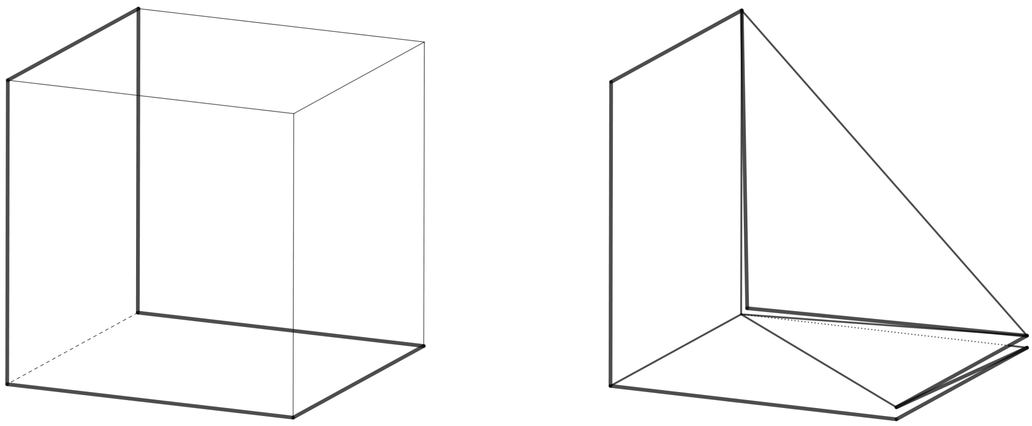

Figure 23: another PEO covering six edges of a cube

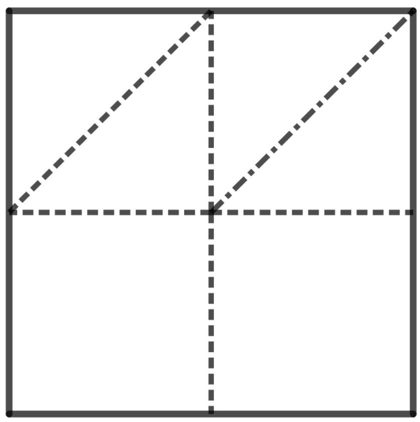

Figure 24: folding pattern

6.2 8 Edges on a Cube

We now turn our attention to folding a unit square in such a way that the edges of the resulting origami are concurrent with eight edges of a cube with edge-length $\frac{1}{2}$, without repetition. To this purpose, we first note that any possible origami O^* with this property must leave the edge of the square as shown in Figure 25. This placement is unique, except for rotations, as there are three cube edges meeting in each cube vertex, and any edge of a possible O^* must pass through any such vertex an even number of times. This means that the edge of the O^* will pass through any cube vertex twice, as eight of the twelve cube edges must be covered.

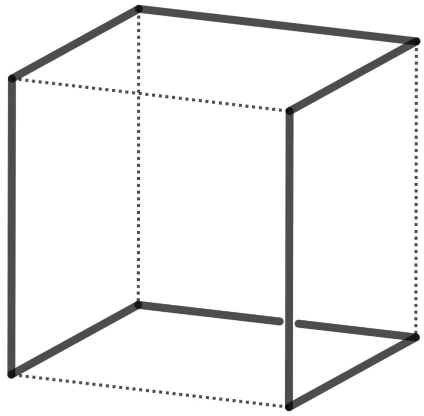

Figure 25: 8 edge placement

In order to fold an O^* with edges as shown in Figure 25, the four corners of the folding square must be placed in four vertices of the cube. Such a placement in one cube vertex is suggested in Figure 27.

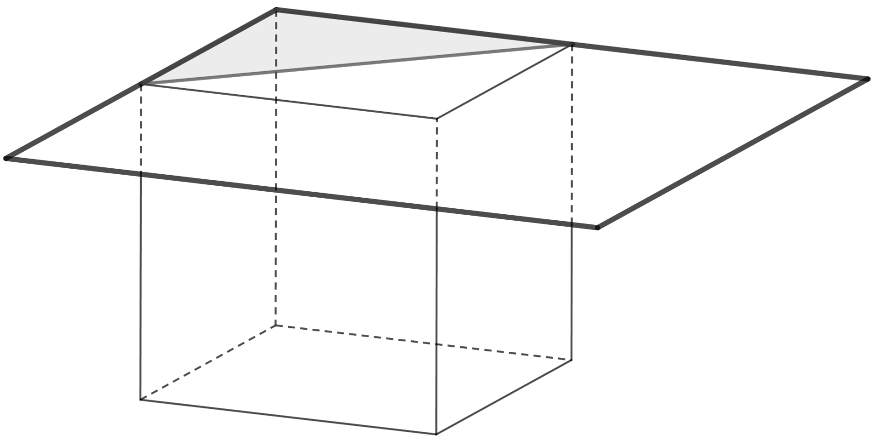

Figure 26: square on cube 1

If we name the cube vertices A,B,C,D,E,F,G,H in the order required by the edge polyline shown in Figure 25, we see that triangle AHB must be covered by triangle $A'H'B'$ on the folding square $A'G'E'C'$, as illustrated in Figure 27. Specifically, this means that there can be no additional creases in the folding square that cross this triangle, as any such a crease would shorten the distance between at least one pair of triangle vertices, and these right triangles must both have legs of equal length $\frac{1}{2}$, and must therefore be congruent.

Moving on to Figure 28, we then see that this must also hold true for triangles BCD, DEF and FGH, which must be covered by the congruent triangles $B'C'D'$, $D'E'F'$ and $F'G'H'$, respectively. Lines $B'D'$, $D'F'$, $F'H'$ and $H'B'$ of the folding square must therefore fold to diagonals BD, DF, FH and HB of the cube faces, respectively. In other words, the sides of square $B'D'F'H'$ must fold to four edges of the regular tetrahedron $BDFH$.

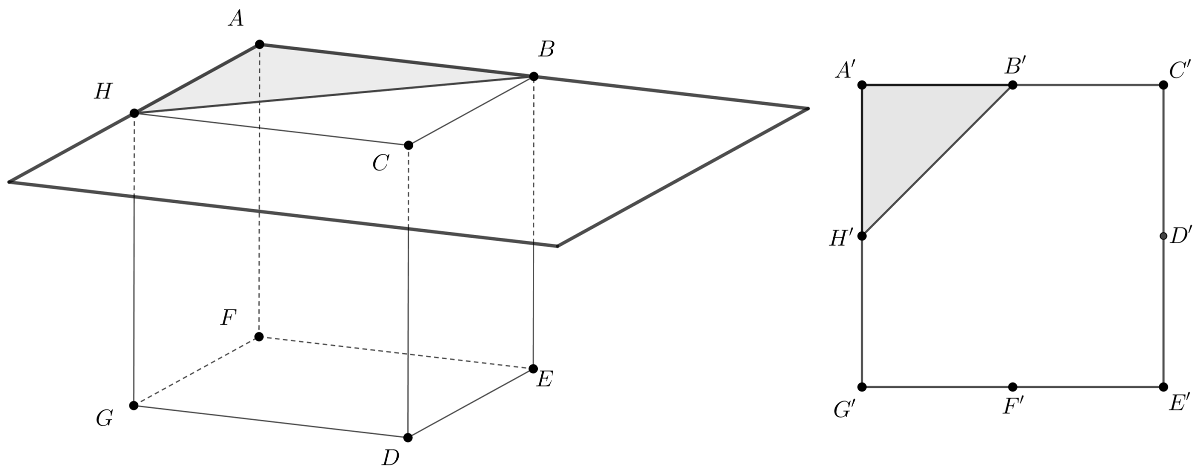

Figure 27: square on cube 2

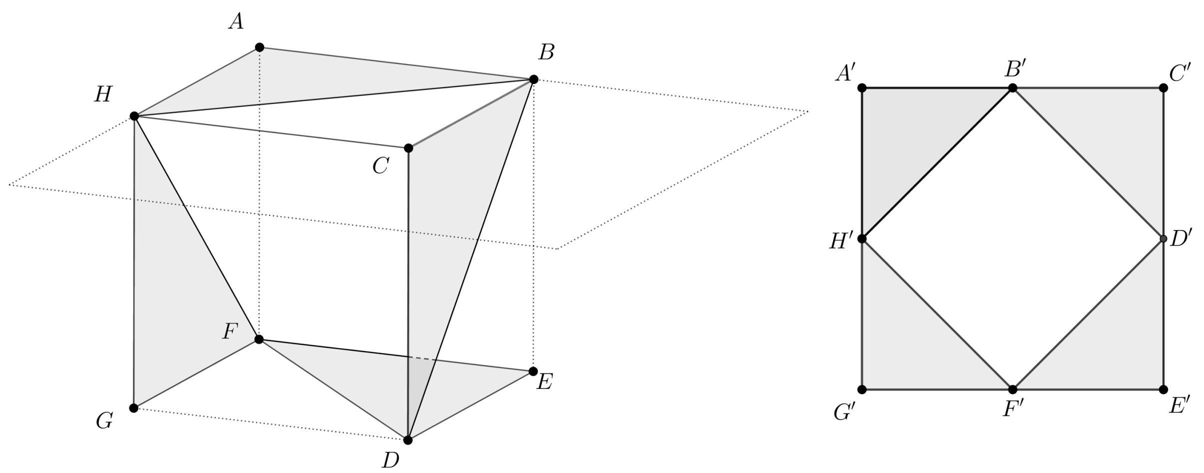

Figure 28: square on cube 3

We already have a method to do this at our disposal from Section 5, and this means that we can create an origami O^* with the required property as shown in Figure 29 (with the embedded tetrahedron suggested in the right-hand part of the figure), applying the folding pattern in Figure 30.

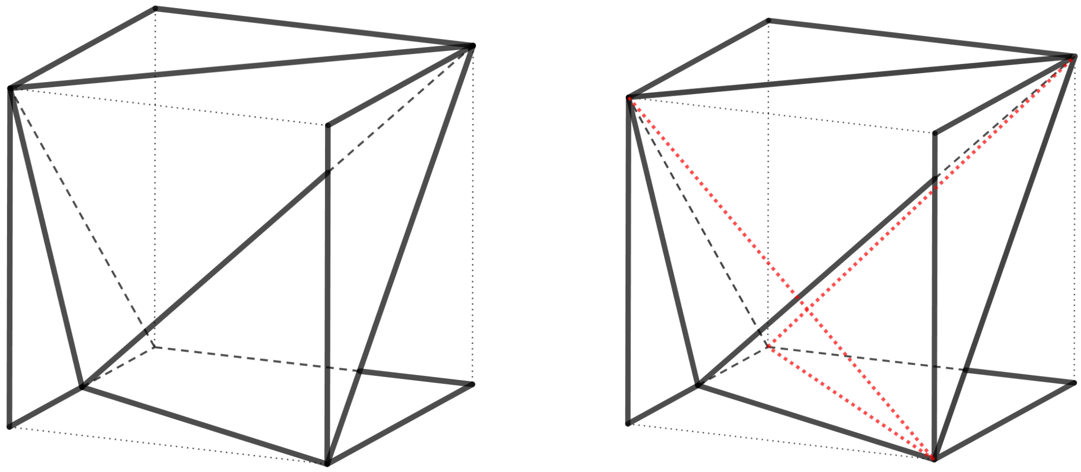

Figure 29: a PEO covering eight edges of a cube

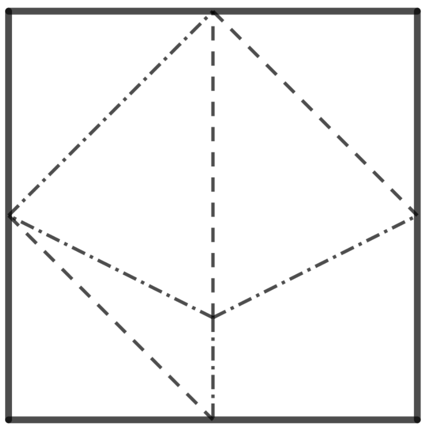

Figure 30: folding pattern

6.3 12 Edges on a Cube

After discussing the above models, it is intriguing to consider what the simplest PEO covering all twelve edges of the cube might look like. Unfortunately, no solution to this problem is available at this time, but perhaps an interested reader of this paper might be motivated to find such a model. It is clear that there can be no such model with edge-lengths of $\frac{1}{3}$, since any such a model must result from three cube edges being covered by each of the sides of the folding square. This would require a closed polyline composed of all twelve edges of the cube, each covered exactly once. Such a polyline is not possible, since each closed polyine on the edges of the cube must have an even number of segments meeting in each of the cube's vertices. Since an odd number of edges, namely three, meet in each vertex, such a polyline cannot exist. It does seem like there could be a reasonably simple PEO with an edge length of $\frac{1}{4}$, however, and finding an example of such a model would seem to be a logical next step.

6.4 8 Edges on an Octahedron

Having discovered a Type (E) PEO on eight edges of the cube, we can optimistically hope that such a model might also be reasonable easy to find on eight edges of the regular octahedron. As it happens, this does, indeed, turn out to be the case.

First of all, we will once more need to find a closed polyline on eight edges on the regular octahedron, and such a polyline is illustrated in Figure 31.

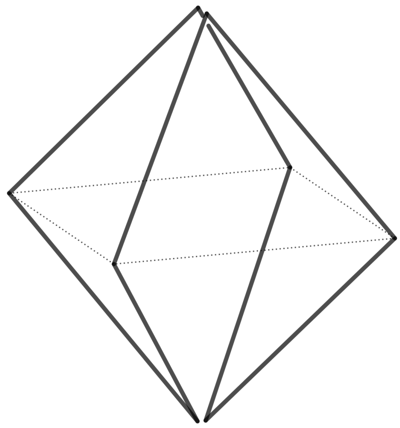

Figure 31: closed polyline on 8 octahedron edges

We start by folding the traditional square base and opening it up a bit, such that the four flaps are perpendicular to each other. This is already quite suggestive of a regular octahedron, as we see in Figure 32.

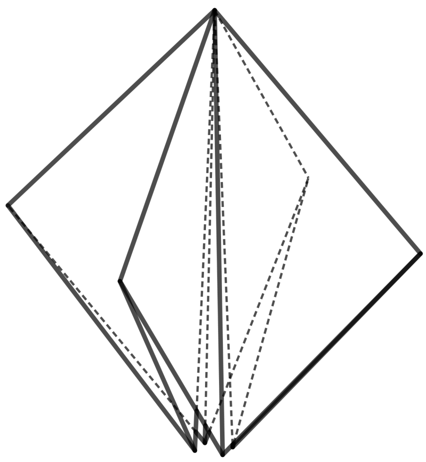

Figure 32: square base with perpendicular flaps

To complete the PEO, we now only need to fold two of the individual bottom flaps up, as shown in Figure 33. This results in a model with its edge on eight octahedron edges as shown in Figure 34.

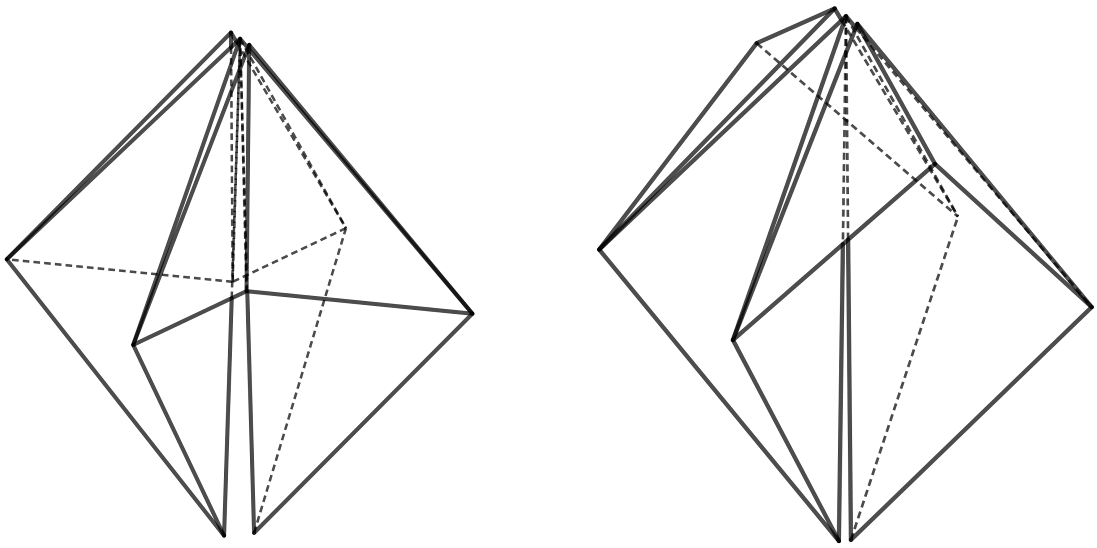

Figure 33: 8 edges

We note that there are two distinct ways to place the resulting flaps, with the version shown on the left having these flaps pushed inward toward the center of the model, and the version shown on the right with these flaps sticking out.

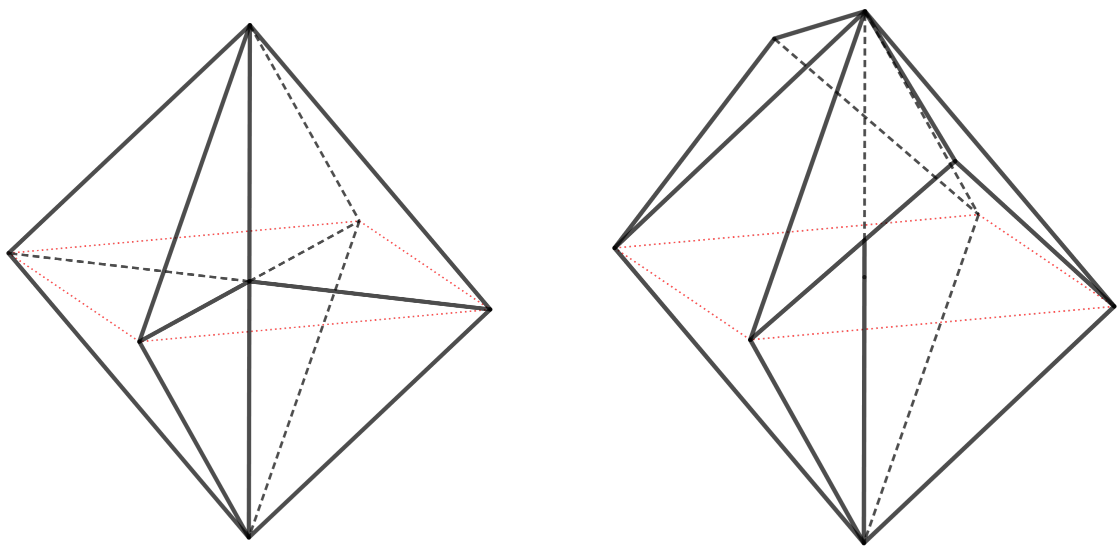

Figure 34: 8 edges

Finally, the folding patterns for these two versions of the model are shown in Figure 35.

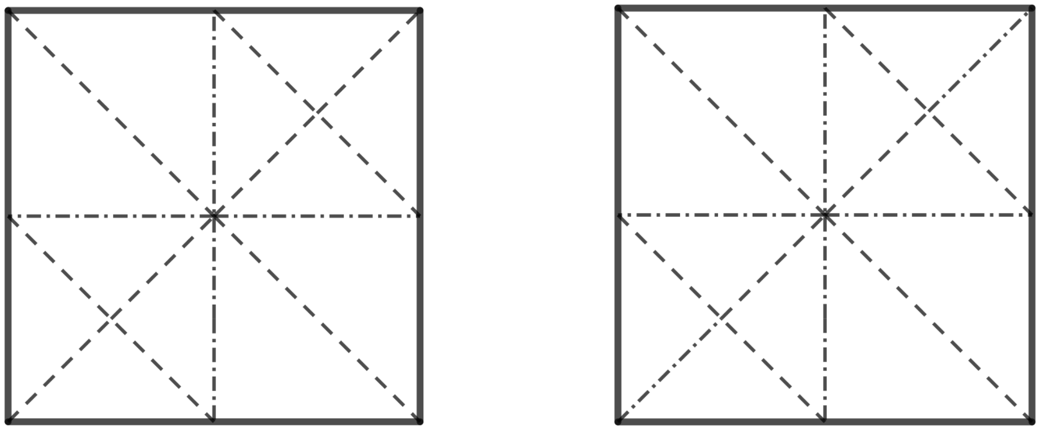

Figure 35: 8 edges

6.5 12 Edges on an Octahedron

Again, it is now intriguing to consider what the simplest PEO covering all twelve edges of the octahedron might look like. Unfortunately, as for the cube, no solution to this problem is currently available. In the case of the octahedron, there may well be a model with edge-lengths of $\frac{1}{3}$, since a closed polyline composed of all twelve edges of the octahedron, with each edge covered exactly once, does exist. Finding a model covering such a polyline could also be a worthwhile next step in the discovery of interesting PEOs.

7 Conclusion

The models described in this paper are not too complex, but they all share the fascinating property of having their edges exclusively on the edges of Platonic solids. As mentioned in the introduction, we have been searching for models determined by their edges, and we have found a number of models determined by parts of the edge networks of cubes, tetrahedra and octahedra. Of course, we have not yet found any PEOs that are reconstructions of the entire edge network of a Platonic solid, apart from the trivial reconstruction we can derive from the one-edge models in Section 4. Finding such models would certainly be a fascinating next step!

Bibilography

[1] Geretschläger, R. and Keeling, S.L., *Using Direct and Constructive Methods for the Existence of Origami Models with Given Boundary Conditions*, Espacio Matemático Vol. 1 (No. 1) 2020, (ISSN: 2711-1792), pp. 54-71,
https://revistas.uan.edu.co/index.php/espaciomatematico/article/view/1626/1246